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# GENERATING SERIES TECHNIQUES FOR COMPUTING DARMON–DASGUPTA UNITS OVER REAL QUADRATIC FIELDS

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Abstract. Taking advantage of recent progress on Hilbert's twelfth problem, we describe a new algorithm to compute $p$-units in abelian extensions of real quadratic fields. This algorithm builds on recent work of Charollois which provides formulas for the $p$-adic interpolation of special values of Lerch's cotangent zeta function. These special values are given as coefficients of products of elementary generating series. Using the FLINT Library for fast computations of these coefficients, we are able to compute these $p$-units to 1000 digits of $p$-adic precision for small primes $p$. We are also able to do these computations for primes $p$ up to 600. This is an order of magnitude better than previous works.

## Contents

## 1. Introduction

Let $F$ be a real quadratic field, let $p$ be a prime number inert in $F$ and $\ell \geq 3$ be an auxiliary prime different from $p$. In 2006, Darmon and Dasgupta [DD06] constructed a $p$-adic invariant $u_\ell(\omega)$ attached to a real quadratic irrationality $\omega \in F$. The $p$-adic number $u_\ell(\omega)$ is defined as a multiplicative integral with respect to a $p$-adic measure. They formulated an algebraicity conjecture for this $p$-adic invariant, predicting that $u_\ell(\omega)$ is a $p$-unit in the narrow ring class field $H_\omega$ of $\mathbb{Q}(\omega)$. Their explicit formula for $u_\ell(\omega)$ makes it a genuine refinement of the Gross–Stark conjecture. Taking advantage of this description, Dasgupta deduces in [Das07] an algorithm to compute this $p$-unit. More precisely, they introduce a $p$-adic measure $\mu_\ell^{DD}$ on $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ such that

$$\log_p(u_\ell(\omega)) = \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x - \omega y) d\mu_\ell^{DD}(x, y), \tag{1}$$

with $\log_p : F_p^\times \to \mathcal{O}_p$ the branch of the logarithm that vanishes at $p$. The field $F_p$ is the completion at $p$ of $F$ and $\mathcal{O}_p$ is its ring of integers. They have explicit formulas to compute this integral (see Subsection 3.2 for more details). The computations of Dasgupta and of Fleischer-Liu [FL21] done for various primes $p \leq 19$ and with 100 digits of $p$-adic precision are in agreement with the conjecture.

This conjecture is now a theorem thanks to the recent groundbreaking work of Dasgupta and Kakde [DK24] (Theorem 1.6) which, together with [Das08] Theorem 8.3, asserts the algebraicity of the $p$-adic number $u_\ell(\omega)$.

Our goal in this paper is to describe a new efficient algorithm to compute the $p$-units $u_\ell(\omega)$. To do so, we will use the results of Charollois [Cha26] relating the values of the Lerch secant zeta function with $p$-adic integrals. These values can be computed as coefficients of generating series which are products of classical generating series for Bernoulli polynomials. For example, one typical generating series we will need to compute is

$$h_\beta(t) = \frac{1}{(e^t - 1)(e^{\beta t} - 1)}, \tag{2}$$

where $\beta \in \mathbb{Q}(\omega)$ is a unit in $\mathbb{Z}[\omega]$ of norm 1. This generating series has been studied by Lerch in [Ler04]; Lerch states that for $k \in \mathbb{Z}_{\geq 0}$ the $2k$-th coefficient is given by

$$\frac{(-1)^k}{1 - \beta^{2k}} h_\beta(t)[t^{2k}] = \sum_{n=1}^{+\infty} \frac{\cot(n\beta\pi)}{(2n\pi)^{2k+1}}. \tag{3}$$

Here, $h_\beta(t)[t^{2k}]$ denotes the coefficient of $t^{2k}$ in the generating series $h_\beta(t)$. Following [Cha26], we will introduce a generating series $h_\omega^{(\ell)}$ which is a sum of generating series similar to $h_\beta(t)$. We prove in Theorem 6 that the coefficients of $h_\omega^{(\ell)}$ encode moments of the form

$$(1 - p^m) m!\, h_\omega^{(\ell)}(t)[t^m] = \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x + \omega y)^m d\mu_\ell^{DD}(x, y), \tag{4}$$

where $m \in \mathbb{Z}_{\geq 0}$ is even. Using the FLINT library [FLI25], we are able to calculate efficiently these moments as coefficients of the generating series $h_\omega^{(\ell)}$. We give an algorithm to compute $\log_p(u_\ell(\omega))$ using these moments by extracting coefficients of $h_\omega^{(\ell)}$. This will allow us to compute the $p$-unit $u_\ell(\omega)$ for primes $p$ up to 600 in favorable cases. For smaller primes $p$ such as $p = 7$ or $p = 11$, this strategy also allows us to compute with very high accuracy, such as a thousand $p$-adic digits, a significant improvement in the computational aspect of Hilbert's twelfth problem for real quadratic fields.

We now describe the structure of this paper. First, we recall the strategy to interpolate Bernoulli numbers as this can be considered as the one dimensional case of our method. Next, we introduce the generating series $h_\omega^{(\ell)}$. In Subsection 2.3, we give an algorithm to compute its coefficients. Theorem 6 relates these coefficients with moments of the measure $\mu_\ell^{DD}$. Using Proposition 8, we give an algorithm in Section 3 to compute $u_\ell(\omega)$. To do so, we relate the $p$-adic integral appearing in Equation (1) to coefficients of multiple generating series similar to $h_\omega^{(\ell)}$ (see Equation (40)). In Propositions 11 and 12, we compare the complexities of our algorithm and Dasgupta's algorithm, showing improvements with respect to both the size of the prime $p$ and the precision parameter $M$. Finally in Section 4, we give tables of examples of computations of several $u_\ell(\omega)$ that live in the narrow Hilbert class field of $F$ for various primes $p \leq 600$ and discriminant $D$.

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## 2. Generating series

### 2.1. Kummer's congruence for Bernoulli numbers.

As our approach will bear a resemblance to the strategy to $p$-adically interpolate Bernoulli numbers, we give a quick summary of the steps involved. Recall the definition of the Bernoulli polynomials $b_k(z)$ using the following generating series

$$f_x(z) := \frac{e^{xz}}{e^x - 1} = \sum_{k \geq 0} b_k(z) x^{k-1}. \tag{5}$$

Here, $k!\, b_k(z) = B_k(z)$, where $B_k(z)$ denotes the usual $k$-th Bernoulli polynomial. Following [Kob84] Chapter II, a way to $p$-adically interpolate the Bernoulli numbers $b_k(0)$ is as follows. Attached to an odd prime number $p$, one can define a distribution $\mu_{b,1}$ on $\mathbb{Z}_p$ by:

$$\mu_{b,1}(a + p^r\mathbb{Z}_p) = b_1\left(\frac{a}{p^r}\right), \tag{6}$$

for any $0 \leq a \leq p^r - 1$ and $r \in \mathbb{Z}_{\geq 0}$. This distribution is not a measure as it is not bounded when $r$ grows large. To deal with this, we will do a smoothing. Let $\ell$ be a prime number different from $p$ and define the smoothed Bernoulli distribution on a compact open subset $U$ of $\mathbb{Z}_p$ as

$$\mu_{b,1}^{(\ell)}(U) = \mu_{b,1}(U) - \ell^{-1}\mu_{b,1}(\ell U). \tag{7}$$

The distribution $\mu_{b,1}^{(\ell)}$ is now a measure on $\mathbb{Z}_p$ (see [Kob84] Subsection 2.5). For $k \in \mathbb{Z}_{\geq 0}$, we have a particularly interesting relation between $\mu_{b,1}^{(\ell)}$ and $b_k(0)$:

$$-(1 - p^k)k!\, b_{k+1}(0) = \frac{1}{\ell^{-k-1} - 1} \int_{\mathbb{Z}_p - p\mathbb{Z}_p} x^k d\mu_{b,1}^{(\ell)}(x). \tag{8}$$

Note the independence of $\ell$ on the left-hand side of the equality. The right-hand side allows us to define the value of the Kubota–Leopoldt zeta function on negative integers:

$$\zeta_p(1 - k) = -(1 - p^{k-1})(k-1)!\, b_k(0). \tag{9}$$

Let's now remark that the generating series

$$f_x(0) = \sum_{k=0}^{+\infty} b_k(0)x^{k-1} \in \frac{1}{x}\mathbb{Q}[[x]]$$

is a generating series with rational coefficients such that for any prime number $p$, the quantity

$$(1-p^k)k!\, f_x(0)[x^k]$$

can be $p$-adically interpolated when $k$ varies. More precisely, we have the following special case of Kummer's congruence.

**Proposition 1** (Kummer). *Let $m, m' \in \mathbb{Z}$, $N \geq 0$, $k = m(p-1)p^N$ and $k' = m'(p-1)p^N$. Then*

$$(1-p^k)k!\, b_{k+1}(0) \equiv (1-p^{k'})k'!\, b_{k'+1}(0) \bmod p^{N+1}.$$

The following construction can be considered as a 2-dimensional case of this subsection.

2.2. **Product of generating series.** For any $z \in \mathbb{R}$, define its fractional part $\{z\}$ by

$$0 \leq \{z\} < 1 \text{ and } z - \{z\} \in \mathbb{Z}.$$

Let $c > 0, d$ be two coprime integers and $v = (v_1, v_2) \in \mathbb{Q}^2$. We now define the generating series in two variables:

$$g(c,d,v)(x,y) \coloneqq \sum_{j=0}^{c-1} f_x\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_y\left(\frac{j+\{v_2\}}{c}\right) \in \frac{1}{xy}\mathbb{Q}[[x,y]]. \tag{10}$$

Let $\omega$ be a real quadratic number and $\beta = c\omega + d$. We also define

$$h_\omega(t,c,d,v) \coloneqq g(c,d,v)(t,\beta t) \in \frac{1}{t^2}\mathbb{Q}(\omega)[[t]]. \tag{11}$$

**Remark 2.** *When $v = (0,0)$ and $(c,d) = (1,0)$, we are in the case considered by Lerch in Equations (2) and (3).*

*When $v = (0, \frac{3}{4})$, Charollois–Greenberg proved a relation similar to Equation (3), see [CG14] Equation (5). For suitable $(c,d)$, they prove that for all $k \in \mathbb{Z}_{\geq 1}$, there is an explicit constant $C_k \in \mathbb{Q}$ such that*

$$\frac{1}{1-\beta^{2k-1}} h_\omega(t,c,d,(0,\frac{3}{4}))[t^{2k-1}] = C_k \sum_{n=1}^{+\infty} \frac{\sec\left(n\frac{\omega}{2}\pi\right)}{(2\pi n)^{2k}}.$$

*They deduce that the right-hand side of the equality lives in $\mathbb{Q}(\omega)$, as conjectured by Lalín, Rodrigue and Rogers in [LRR14].*

2.3. **Fast computations of coefficients of products of generating series.** The coefficients of $h_\omega$ (and of its smoothed version $h_\omega^{(\ell)}$ introduced below, see Equation (15)) will be our main interest in this paper. They will also be the main computational challenge when it comes to computing Darmon–Dasgupta units with our method. In this subsection we explain our strategy to compute efficiently the coefficients of $h_\omega$. Recall that for some given integers $c > 0, d$, and a vector $v = (v_1, v_2) \in \mathbb{Q}^2$,

$$\begin{aligned} h_\omega(t,c,d,v) &= \sum_{j=0}^{c-1} f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_{\beta t}\left(\frac{j+\{v_2\}}{c}\right) \\ &= \sum_{j=0}^{c-1} \frac{e^{t\{v_1-\frac{d}{c}(j+v_2)\}}}{(e^t-1)} \frac{e^{\beta t\left(\frac{j+\{v_2\}}{c}\right)}}{(e^{\beta t}-1)}, \end{aligned} \tag{12}$$

where $\beta = c\omega + d \in \mathbb{Q}(\omega)$. A priori, this is a generating series with coefficients in $\mathbb{Q}(\omega)$. However, in our implementation, we perform as many computations as possible over $\mathbb{Q}$. Our problem boils down to computing quickly the coefficients of

$$f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_{\beta t}\left(\frac{j+\{v_2\}}{c}\right)$$

for $0 \le j \le c-1$. To do so, we will call the FLINT [FLI25] library using SageMath [The22]. FLINT has a routine which allows us to compute quickly the product and inverses of generating series with rational coefficients.

To compute $h_\omega(t,c,d,v)$, we will compute the generating series with rational coefficients

$$t f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) = \frac{t e^{t\{v_1-\frac{d}{c}(j+v_2)\}}}{e^t-1} \quad \text{and} \quad t f_t\left(\frac{j+\{v_2\}}{c}\right) = \frac{t e^{t\frac{j+\{v_2\}}{c}}}{e^t-1}.$$

Note that we multiplied by $t$ both generating series so $\frac{e^t-1}{t} \in \mathbb{Q}((t))$ is in fact invertible in $\mathbb{Q}[[t]]$. Then, we only need to do one last sum of elements of $\mathbb{Q}(\omega)$ :

$$\begin{aligned} &\left[f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_{\beta t}\left(\frac{j+\{v_2\}}{c}\right)\right][t^n] \\ &= \sum_{k=-1}^{n+1} \beta^k f_t\left(\frac{j+\{v_2\}}{c}\right)[t^k] f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right)[t^{n-k}]. \end{aligned} \tag{13}$$

This gives us the following algorithm to compute the coefficients of $h_\omega$.

**Algorithm 1:** Computing the coefficients of $h_\omega(t,c,d,v)$

```
Input: An integer $n$, a real quadratic number $\omega$, two integers $c > 0, d$, a
       vector $v = (v_1, v_2) \in (\mathbb{Q}/\mathbb{Z})^2$.
Output: The coefficient $h_\omega(t,c,d,v)[t^n]$.
Compute $\frac{e^t-1}{t}$ up to $t^{n+1}$;
Compute $\left(\frac{e^t-1}{t}\right)^{-1}$ up to $t^{n+1}$;
for $j = 0, .., c-1$ do
    Compute the generating series $t f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right)$ and $t f_t\left(\frac{j+\{v_2\}}{c}\right)$
     up to $t^{n+1}$;
    Compute $f_t\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_{\beta t}\left(\frac{j+\{v_2\}}{c}\right)[t^n]$ using Equation (13);
return $h_\omega(t,c,d,v)[t^n]$
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The library FLINT [FLI25] also allows us to compute generating series with coefficients in $\mathbb{Q}(\omega)$, however staying in $\mathbb{Q}$ for most steps makes for faster computations.

We describe the strategy used by FLINT for these calculations. First, we need to be able to compute the inverse of the generating series

$$\frac{e^t-1}{t}.$$

This is done using Newton iteration. Then, the product of polynomials to compute $f_t$ is done using fast Fourier transform (FFT) for polynomials of high degree. Otherwise, it is done using linear recurrence. See [Ber08] for more details. The FFT and the computation of the inverse up to $O(t^M)$ are done in $O(M\log(M))$ computations in $\mathbb{Q}$. The coefficients of the generating series grows as Bernoulli numbers in $(O(M\log(M)))$. The time complexity to compute the $M$ first coefficients is $O(M^2\log(M)^2)$. From there, it is the same complexity to compute $h_\omega(t,c,d,v)[t^k]$ for all $k\leq M$.

We now explain how the coefficients of a smoothed version of $h_\omega$ can be understood as $p$-adic integrals.

2.4. **$p$-adic measures on $\mathbb{Z}_p^2$.** For any integer $M\geq 1$, the generating series $g(c,d,v)(x,y)$ satisfies the distribution relation

$$\sum_{i,j=0}^{M-1} g\left(c,d,\left(\frac{v_1+i}{M},\frac{v_2+j}{M}\right)\right)(x,y)=g(c,d,v)\left(\frac{x}{M},\frac{y}{M}\right). \tag{14}$$

We will deduce that the coefficients of $g$ have a $p$-adic interpretation as integrals on $\mathbb{Z}_p^2-p\mathbb{Z}_p^2$. To do so, as it was the case with the Bernoulli numbers in Subsection 2.1, we introduce a smoothed version of this generating series. Let $c>0$ be an integer divisible by a prime number $\ell\geq 3$, and let $d$ be an integer coprime to $c$. Define

$$g^{(\ell)}(c,d,v)(x,y):=g\left(\frac{c}{\ell},d,(\ell v_1,v_2)\right)\left(\frac{x}{\ell},\frac{y}{\ell}\right)-\ell g(c,d,v)(x,y)$$

and

$$h_\omega^{(\ell)}(t,c,d,v):=h_{\ell\omega}\left(\frac{t}{\ell},\frac{c}{\ell},d,(\ell v_1,v_2)\right)-\ell h_\omega(t,c,d,(v_1,v_2)). \tag{15}$$

**Proposition 3.** *The generating series $g^{(\ell)}(c,d,v)(x,y)$ is regular at $(x,y)=(0,0)$. Furthermore, if $\ell\geq 5$, then $g^{(\ell)}(c,d,v)(0,0)\in\mathbb{Z}$ and $g^{(\ell)}(c,d,v)(0,0)\in\frac{1}{3}\mathbb{Z}$ if $\ell=3$.*

*Proof.* This proposition is a special case of Theorem 4 in [CDG15]. Using their notation, we are in the case where $A=\begin{pmatrix} d & * \\ c & * \end{pmatrix}\in\Gamma_0(\ell)$, $n=2$, $P=1$ and $Q\in\mathbb{R}^2$ a suitable sign-vector. For more details, see [Cha26] Proposition 2.3. ☐

Let $p$ be an odd prime number. We define a measure $\mu_\ell(c,d)$ on $\mathbb{Z}_p^2-p\mathbb{Z}_p^2$ as follows.

**Definition 4.** For $(a,b)\in\mathbb{Z}_p^2-p\mathbb{Z}_p^2$ and $r\geq 1$, let

$$U_{a,b,r}:=(a,b)+p^r\mathbb{Z}_p^2. \tag{16}$$

Let $\ell \neq p$ be an odd prime number, let $c \in \mathbb{Z}_{>0}$ be divisible by $\ell$, and let $d \in \mathbb{Z}$ be prime to $c$. We define the value of $\mu_\ell(c,d)$ on the compact open subset $U_{a,b,r}$ by

$$\mu_\ell(c,d)(U_{a,b,r}) := \delta_\ell g^{(\ell)}\left(c, d, \left(\frac{a}{p^r}, \frac{b}{p^r}\right)\right)(0,0) = \delta_\ell h_\omega^{(\ell)}\left(t, c, d, \left(\frac{a}{p^r}, \frac{b}{p^r}\right)\right)[t^0], \tag{17}$$

where

$$\delta_\ell = \begin{cases} \ell & \text{if } \ell = 3 \\ 1 & \text{if } \ell \geq 5. \end{cases} \tag{18}$$

**Lemma 5.** *The assignment*

$$U_{a,b,r} \longmapsto \mu_\ell(c,d)(U_{a,b,r})$$

*defines a $\mathbb{Z}$-valued measure on $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$.*

*Proof.* The sets $U_{a,b,r}$ form a basis of compact open subsets of $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$. To show that $\mu_\ell(c,d)$ is a $p$-adic distribution, it is enough to check that for all $(a,b) \in [0,p-1]^2 \setminus \{0,0\}$ and $r \geq 0$,

$$\mu_\ell(c,d)(U_{a,b,r}) = \sum_{i,j=0}^{p-1} \mu_\ell(c,d)(U_{a+ip^r, b+jp^r, r+1}). \tag{19}$$

This equality comes from the distribution relation (14) satisfied by the generating series $g(c,d,(a/p^r, b/p^r))$ and its smoothed version $g^{(\ell)}(c,d,(a/p^r,b/p^r))$. The fact that $\mu_\ell(c,d)$ is $\mathbb{Z}$-valued is a immediate consequence of Proposition 3.

□

We can write explicitly the right-hand side of Equation (17) as Dedekind sums. Let $\tilde{b}_1$ be the 1-periodic function equal to $b_1$ on $[0,1[$. We define the Dedekind sum for each matrix $\sigma \in M_2(\mathbb{Z})$ and $r,s \in \mathbb{Z}_{\geq 0}$ by

$$\mathcal{D}_{r,s}(\sigma, v) := \sum_{x \in \mathbb{Z}^2/\sigma\mathbb{Z}^2} \mathcal{B}_{r,s}(\sigma^{-1}(x+v)), \tag{20}$$

where for $y = (y_1, y_2) \in \mathbb{Q}$, $\mathcal{B}_{r,s}(y) = r!\, s!\, \tilde{b}_r(y_1)\tilde{b}_s(y_2)$. We have

$$\mu_\ell(c,d)(U_{a,b,r}) = \mathcal{D}_{1,1}(\sigma(c/\ell, d), (\ell v_1, v_2)) - \ell\mathcal{D}_{1,1}(\sigma(c,d), v), \tag{21}$$

where $(v_1, v_2) = \left(\frac{a}{p^r}, \frac{b}{p^r}\right)$ and $\sigma(c,d) = \begin{pmatrix} 1 & d \\ 0 & c \end{pmatrix}$.

The following theorem relates the coefficients of $h_\omega^{(\ell)}$ with specific moments of this measure.

**Theorem 6.** *Let $c \geq 1$ be an integer divisible by a prime $\ell \geq 3$ and $d \in \mathbb{Z}$ prime to $c$, let $\omega$ be a real quadratic number. Let $p \neq \ell$ be a prime number. For any integer $m \geq 0$, for $(a,b) \not\equiv (0,0) \bmod p$, we have the following equality in $\mathbb{Q}(\omega)$.*

$$p^m m!\, h_\omega^{(\ell)}\left(t, c, d, \left(\frac{a}{p}, \frac{b}{p}\right)\right)[t^m] = \int_{(a,b)+p\mathbb{Z}_p^2} (x+\omega y)^m d\mu_\ell(c,d)(x,y). \tag{22}$$

*When $(a,b) = (0,0)$, we have*

$$(1-p^m) m!\, h_\omega^{(\ell)}(t,c,d,(0,0))[t^m] = \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x+\omega y)^m d\mu_\ell(c,d)(x,y). \tag{23}$$

*Proof.* Equation (22) is a consequence of Theorem 4.2 in [CD14]. Using the notation of [CD14], we are in the case $n = 2$, $P = (x + \omega y)^m$, $M = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, $v = (0, 0)$ and $Q$ is an appropriate sign vector. For more details, see [Cha26] Theorem 2.7. We deduce Equation (23) from Equation (22) as follows. From the distribution property of $h_\omega^{(\ell)}$ (implied by Equation (14)), we have

$$(1 - p^m) h_\omega^{(\ell)}(t, c, d, (0, 0))[t^m] = p^m \sum_{\substack{(a,b) \bmod p \\ a,b \neq 0,0}} h_\omega^{(\ell)}(t, c, d, (a/p, b/p))[t^m]. \tag{24}$$

We complete the proof by multiplying by $m!$ and using Equation (22). □

Writing these coefficients as integrals gives us the following congruences between them.

**Corollary 7.** *Let $p \geq 3$ be a prime number inert in $\mathbb{Q}(\omega)$ such that $(\omega, p) = 1$. Let $r, k, k' \in \mathbb{Z}_{\geq 0}$, such that $k \equiv k' \bmod (p^2 - 1)p^r$, then*

$$(1 - p^k) k! \, h_\omega^{(\ell)}(t, c, d, (0, 0))[t^k] \equiv (1 - p^{k'}) k'! \, h_\omega^{(\ell)}(t, c, d, (0, 0))[t^{k'}] \bmod p^{r+1}.$$

*Proof.* Let $(x, y) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$, then $x + \omega y \neq 0 \bmod p$. Indeed, $p$ is inert in $\mathbb{Q}(\omega)$ so $\overline{\omega} \in \mathbf{F}_{p^2} - \mathbf{F}_p$. In $\mathbf{F}_{p^2}$, we have the congruence $(x + \omega y)^{p^2 - 1} \equiv 1 \bmod p$. Using the fact that for any $z \in 1 + p\mathcal{O}_p$, if $n, n'$ are two integers congruent modulo $p^r$, then

$$z^n \equiv z^{n'} \bmod p^{r+1}.$$

We deduce that

$$(x + \omega y)^k \equiv (x + \omega y)^{k'} \bmod p^{r+1}.$$

The measure $\mu_\ell(c, d)$ has values in $\mathbb{Z}$ so

$$v_p\left( \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x + \omega y)^k d\mu_\ell(c, d) - \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x + \omega y)^{k'} d\mu_\ell(c, d) \right) \geq r + 1.$$

Using Theorem 6 we are done. □

The case of most interest for us will be when $k \equiv 0 \bmod (p^2 - 1)p^r$, note the resemblance to Proposition 1. By taking this limit as $k$ goes to 0 $p$-adically, we are able to recover the integral of the log of the linear form $x + \omega y$ (see Remark 10). We now explain how this integral relates to the unit $u_\ell(\omega)$ introduced before.

### 2.5. Description of the Darmon–Dasgupta unit using generating series.

Let $\omega$ be a real quadratic number, satisfying the quadratic equation

$$A\omega^2 + B\omega + C = 0,$$

with integers $A, B$ and $C$ with $\gcd(A, B, C) = 1$, where $A > 0$ is divisible by a prime number $\ell \geq 3$. Let $\gamma_\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the generator of $\mathrm{Stab}_{SL_2(\mathbb{Z})}(\omega)$ such that $\beta = c\omega + d > 1$. Let $p \neq \ell$ be a prime inert in $\mathbb{Q}(\omega)$. The $p$-adic logarithm of the Darmon–Dasgupta unit $u_\ell(\omega)$ is given by the following (see [Cha26] Theorem 2.9).

**Proposition 8.** *With the notation as above,*

$$\delta_\ell \log_p(u_\ell(-\omega)) = 12 \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x + \omega y) d\mu_\ell(c, d)(x, y), \tag{25}$$

*where $\delta_\ell = 3$ if $\ell = 3$, and $\delta_\ell = 1$ if $\ell \geq 5$.*

*Proof.* We reproduce the proof of Charollois for completeness. We want to make use of the formula (32) of [Das07]:

$$\log_p(u_\ell(-\omega)) = \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y) d\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(x,y). \tag{26}$$

Here $\mu_\ell^{DD}$ is the measure on $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ described in [Das07] Proposition 3.2.

$$\begin{aligned}\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(U_{a,b,r}) = 12\Bigg[&\sum_{j=0}^{c/\ell-1} b_1^\sharp\left(\ell v_1 - d\frac{(j+v_2)}{c/\ell}\right) b_1^\sharp\left(\frac{j+v_2}{c/\ell}\right) \\ &- \ell\sum_{j=0}^{c-1} b_1^\sharp\left(v_1 - d\frac{(j+v_2)}{c}\right) b_1^\sharp\left(\frac{j+v_2}{c}\right)\Bigg],\end{aligned} \tag{27}$$

where $b_1^\sharp$ is the 1-periodic function equal to $b_1$ on $]0,1[$ such that $b_1^\sharp(0)=0$. Using their notation, we are in the case where $n_1 = \ell$ and $n_\ell = -1$. We alter the generating series $f_t(z)$ to match this normalization of $b_1^\sharp$:

$$f_t^\sharp(z) \coloneqq \begin{cases} f_t(z) & \text{if } 0 < z < 1 \\ f_t(0) + \frac{1}{2} & \text{if } z = 1. \end{cases} \tag{28}$$

We define $h_\omega^\sharp$ as $h_\omega$ using Equation (11) by substituting $f_t$ for $f_t^\sharp$. The $\ell$-smoothed variant $h_\omega^{\sharp(\ell)}$ is also regular at 0 and we can define a measure $\mu_\ell^\sharp$ as in Definition 4. By construction,

$$\frac{12}{\delta_\ell}\mu_\ell^\sharp(c,d) = \mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}. \tag{29}$$

The generating series $h_\omega^\sharp$ and $h_\omega$ are equal up to an odd function and an additive constant, more precisely,

$$h_\omega^\sharp(t,c,d,(0,0)) - h_\omega(t,c,d,(0,0)) = (\coth(t/2) + \coth(t\beta/2) + 1)/4. \tag{30}$$

In particular, they share the same Taylor coefficient of $t^m$ when $m \geq 2$ is even. This is also the case for their $\ell$-smoothed version. Applying Theorem 6 to both these generating series, we have for all $m \geq 2$ even

$$\begin{aligned}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x+\omega y)^m d\mu_\ell(c,d)(x,y) &= \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x+\omega y)^m d\mu_\ell^\sharp(c,d)(x,y) \\ &= \frac{\delta_\ell}{12}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x+\omega y)^m d\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(x,y).\end{aligned} \tag{31}$$

Let $m = (p^2-1)p^j$ and $j \to +\infty$, both sides go to zero in Equation (31) as the total measure of $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ is zero. Dividing by $\beta^m - 1 = (c\omega + d)^m - 1$ both sides and taking the limit, we have

$$\frac{12}{\log_p(\beta)}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y) d\mu_\ell(c,d)(x,y) = \frac{\delta_\ell}{\log_p(\beta)}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y) d\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(x,y). \tag{32}$$

Let's remark that if $\gamma_\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ is a generator of $\mathrm{Stab}_{SL_2(\mathbb{Z})}(-\omega)$. We can finally invoke Equation (32) in [Das07]. Note that in [Das07], $\ell$ must be divisible by at least three numbers. However, as described in [Das08] Remark 8.2, this construction can still be carried out when $\ell$ is prime. ☐

By the reciprocity law stated in [DD06] Conjecture 2.14, and proved in [DK24] Theorem 6, when the discriminant of $\mathcal{O}_\omega$ is fundamental, we have

$$u_\ell(-\omega) = \overline{u_\ell(\omega)} = u_\ell(\omega)^{-1}. \tag{33}$$

To lighten the notation we will now write

$$u'_\ell(\omega) := u_\ell(-\omega). \tag{34}$$

**Remark 9.** *If we took the quadratic form $Q(x,y) = (x+\omega y)(x+\omega' y) \in \mathbb{Q}[x,y]$ instead of the linear form $(x+\omega y)$, Lemma 4.6 of [DD06] would have given us a weaker form of Proposition 8:*

$$\log_p(\mathrm{Nm}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(u'_\ell(\omega))) = \frac{12}{\delta_\ell}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(Q(x,y))d\mu_\ell(c,d)(x,y). \tag{35}$$

*The left-hand side is also given as the value at 0 of the derivative of a $p$-adic zeta function attached to $F = \mathbb{Q}(\omega)$ (see Equation (80) in [DD06]).*

Our goal for the remainder of this paper is to explain how to compute efficiently the integral given in Equation (25). Using Theorem 6, this boils down to computing certain coefficients of $h_\omega^{(\ell)}$, which can be done efficiently using the methods described in Subsection 2.3.

**Remark 10.** *From the proof of Proposition 8, we see that one strategy for computing $\log_p(u'_\ell(\omega))$ is to express it as the limit*

$$\lim_{\substack{j\to+\infty \\ k=(p^2-1)p^j}} \frac{1}{\beta^k - 1}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} (x+\omega y)^k \, d\mu_\ell(c,d)(x,y) = \frac{\log_p(u'_\ell(\omega))}{\log_p(\beta)}. \tag{36}$$

*Here $k \to 0$ $p$-adically. In particular, using Theorem 6, we would need to compute the $(p^2-1)p^M$-th coefficient of the generating series*

$$h_\omega^{(\ell)}(t,c,d,0,0)$$

*in order to compute $\log_p(u'_\ell(\omega))$ to $M$ digits of $p$-adic precision. However, computing such coefficients becomes very challenging as $M$ grows. Instead, we describe a strategy that requires knowledge only of the first $M + \log(M)$ coefficients of*

$$h_\omega^{(\ell)}\left(t,c,d,\frac{a}{p},\frac{b}{p}\right),$$

*for $(a,b) \in [0,p-1]^2 - \{(0,0)\}$.*

## 3. COMPUTING THE UNITS

### 3.1. An algorithm to compute the $p$-unit.

Using Proposition 8, we want to compute $\log_p(u'_\ell(\omega))$ as the integral:

$$\log_p(u'_\ell(\omega)) = \frac{12}{\delta_\ell}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y)d\mu_\ell(c,d)(x,y) =: \frac{12}{\delta_\ell} I_{\omega,c,d}. \tag{37}$$

By decomposing $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ into residue classes modulo $p$, we obtain:

$$I_{\omega,c,d} = \sum_{(a,b) \bmod p}{}' \int_{U_{a,b}} \log_p(x+\omega y)d\mu_\ell(c,d)(x,y),$$

where the prime indicates that the term $(a,b)=(0,0)$ is omitted, and where $U_{a,b}=(a,b)+p\mathbb{Z}_p^2$. We aim to expand the log around 1 so we write

$$I_{\omega,c,d}=\sum_{(a,b)\bmod p}{}' \int_{U_{a,b}} \log_p\left(\frac{x+\omega y}{a+b\omega}\right) d\mu_\ell(c,d)(x,y)+\log_p(a+b\omega)\mu_\ell(U_{a,b}).$$

Let

$$C_{\omega,c,d}:=\sum_{(a,b)\bmod p}{}' \log_p(a+b\omega)\mu_\ell(c,d)(U_{a,b}).$$

By definition of our measure, $C_{\omega,c,d}$ is given by

$$C_{\omega,c,d}=\sum_{(a,b)\bmod p}{}' \log_p(a+b\omega)h_\omega^{(\ell)}\left(t,c,d,\left(\frac{a}{p},\frac{b}{p}\right)\right)[t^0]. \tag{38}$$

Given $M>0$, we are left to compute

$$I'_{\omega,c,d}=\sum_{(a,b)\bmod p}{}' \int_{U_{a,b}} \log_p\left(\frac{x+\omega y}{a+b\omega}\right) d\mu_\ell(c,d)(x,y)$$

to precision $O(p^{M+1})$. To do so, we use the power series expansion of $\log_p$ around the point 1:

$$\log_p(1+x)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}x^n. \tag{39}$$

Taking into account the denominator in $p$ arising from $\frac{1}{n}$, we take the first $M'=M+\lfloor\log(M)\rfloor$ coefficients in the expansion. More precisely,

$$\begin{aligned}
I'_{\omega,c,d}&=\sum_{(a,b)\bmod p}{}' \int_{U_{a,b}} \log_p\left(\frac{x+\omega y}{a+b\omega}\right) d\mu_\ell(c,d)(x,y)\\
&=\sum_{(a,b)\bmod p}{}' \sum_{n=1}^{M'}\frac{(-1)^{n-1}}{n}\int_{U_{a,b}}\left(\frac{x+\omega y}{a+b\omega}-1\right)^n d\mu_\ell(c,d)(x,y)+O(p^{M+1})\\
&=\sum_{(a,b)\bmod p}{}' \sum_{n=1}^{M'}\frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{(a+b\omega)^j}\int_{U_{a,b}}(x+\omega y)^j d\mu_\ell(c,d)(x,y)+O(p^{M+1}).
\end{aligned}$$

Finally, using Theorem 6, we have

$$I'_{\omega,c,d}=\sum_{(a,b)\bmod p}{}' \sum_{n=1}^{M'}\frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{(a+b\omega)^j}j!\,p^j h_\omega^{(\ell)}(t,c,d,(\frac{a}{p},\frac{b}{p}))[t^j]+O(p^{M+1}). \tag{40}$$

To recognize $u_\ell(\omega)$ knowing $\log_p(u_\ell(\omega))$, we will also need to know $\log_{\beta_p}(u_\ell(\omega))$ for $\beta_p$ a primitive $p^2-1$ root of unity in $F_p$. More precisely, let $\log_{\beta_p}$ be the discrete logarithm with base $\beta_p$

$$\log_{\beta_p}: F_p^\times\to\mathbb{Z}/(p^2-1)\mathbb{Z}.$$

such that

$$\frac{x}{\beta_p^{\log_{\beta_p}(x)}p^{\mathrm{ord}_p(x)}}\in 1+p\mathcal{O}_p \text{ for all } x\in F_p^\times.$$

The expressions for $\log_{\beta_p}(u'_\ell(\omega))$ and $\mathrm{ord}_p(u'_\ell(\omega))$ can also be obtained by extracting appropriate coefficients of generating series. Recall the definition of $\mu^\sharp_\ell(c,d) = \frac{\delta_\ell}{12}\mu^{DD}\{\infty \to \frac{d}{c}\}$ given in Equation (27). Using [Das07] Equation (31), we have

$$\log_{\beta_p}(u'_\ell(\omega)) = \frac{12}{\delta_\ell}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_{\beta_p}(x+\omega y)d\mu^\sharp_\ell(c,d)(x,y). \tag{41}$$

By decomposing the integral over residue classes modulo $p$, we obtain

$$\log_{\beta_p}(u'_\ell(\omega)) = \frac{12}{\delta_\ell}\sideset{}{'}\sum_{(a,b) \bmod p} \log_{\beta_p}(a+b\omega)h^{\sharp(\ell)}_\omega\left(t,c,d,\left(\frac{a}{p},\frac{b}{p}\right)\right)[t^0], \tag{42}$$

where $h^{\sharp(\ell)}_\omega$ is defined analogously to $h^{(\ell)}_\omega$, using the function $f^\sharp_t$ defined in Equation (28). Similarly, by [Das07] Equation (30),

$$\mathrm{ord}_p(u'_\ell(\omega)) = \frac{12}{\delta_\ell}\left(h^{\sharp(\ell)}_\omega(t,c,d,0,0)[t^0] + \frac{\ell-1}{4}\right). \tag{43}$$

The additive factor $\frac{\ell-1}{4}$ arises from the fact that we choose $\ell$ to be prime in the definition of $u'_\ell(\omega)$. More precisely, it is included to match the zeta value defining $\mathrm{ord}_p$, see [Das08] Equation (83).

The computation of the constant term of $h^{\sharp(\ell)}_\omega$ can be carried out using the strategy of Subsection 2.3, replacing $f_t$ by $f^\sharp_t$.

We see that the running time of Algorithm 1 is linear in $c$, the lower-left entry of the stabilizer of $\omega$ in $SL_2(\mathbb{Z})$. Since $c$ can become quite large, we explain how to get rid of this dependence using the cocycle relations satisfied by the measure $\mu_\ell(c,d)$.

As in [Das07] Algorithm 4.1, we write the modular symbol decomposition

$$\left\{\infty \to \frac{d}{c}\right\} = \sum_{\substack{1\le i\le \ell-1\\ \gamma\in\Gamma_0(\ell)}} a_{\gamma,i}\gamma\left\{\infty \to \frac{i}{\ell}\right\} \tag{44}$$

where almost all $a_{\gamma,i}$ are zero. In particular, we can rewrite the integral giving $\log_p(u'_\ell(\omega))$ as

$$\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y)d\mu(c,d)(x,y) = \sum_{i,\gamma} a_{\gamma,i}\int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x - y\gamma^{-1}(-\omega))d\mu(\ell,i)(x,y). \tag{45}$$

Indeed, this identity holds for the measure $\mu^{DD}_\ell$, see [Das07] Equation (45). Using Equation (32) this also holds for our measure $\mu_\ell$. Equation (45) can be rewritten as

$$I_{\omega,c,d} = \sum_{i,\gamma} a_{\gamma,i} I_{\omega_\gamma,\ell,i}, \tag{46}$$

where

$$\omega_\gamma \coloneqq -(\gamma^{-1}(-\omega)). \tag{47}$$

We use the strategy described above to compute these integrals. We use the same decomposition to compute $\log_\beta(u'_\ell(\omega))$ and $\mathrm{ord}_p(u'_\ell(\omega))$.

This gives us the following algorithm to compute $\log_p(u'_\ell(\omega)), \log_{\beta_p}(u'_\ell(\omega))$ and $\mathrm{ord}_p(u'_\ell(\omega))$.

**Algorithm 2:** Computing $\log_p(u'_\ell(\omega))$, $\log_{\beta_p}(u'_\ell(\omega))$ and $\mathrm{ord}_p(u'_\ell(\omega))$ .

```
Input: A real quadratic number $\omega$, a prime number $p$ inert in $\mathbb{Q}(\omega)$, $\ell \geq 3$ a
       prime number different from $p$, $(c,d)$ the lower row of a generator of
       $\mathrm{Stab}_{SL_2(\mathbb{Z})}(\omega)$ such that $c\omega + d > 1$ and an integer $M > 0$.
Output: $\log_p(u'_\ell(\omega))$ with $p$-adic precision $O(p^{M+1})$, $\log_{\beta_p}(u'_\ell(\omega))$ and
        $\mathrm{ord}_p(u'_\ell(\omega))$.
Compute the decomposition $\left\{\infty \to \frac{d}{c}\right\} = \sum_{i,\gamma} a_{\gamma,i}\gamma\left\{\infty \to \frac{i}{\ell}\right\}$;
for $a_{\gamma,i} \neq 0$ do
    for $a = 0,..,p-1$ do
        for $b = 0,..,p-1$ do
            Compute the first $M + \log(M)$ coefficients of
              $h^{(\ell)}_{\omega_\gamma}(t,\ell,i,(\frac{a}{p},\frac{b}{p})) \in \mathbb{Q}(\omega)[[t]]$ using Algorithm 1;
            Compute $h^{\sharp(\ell)}_{\omega_\gamma}(t,\ell,i,(\frac{a}{p},\frac{b}{p}))[t^0]$ using Algorithm 1.
Using the formulas for $C_{\omega_\gamma,\ell,i}$ and $I'_{\omega_\gamma,\ell,i}$ given in Equations (38) and (40),
  compute $I = \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_p(x+\omega y)d\mu_\ell(c,d)(x,y) \bmod p^{M+1}$;
Using Equation (42), compute $\log_{\beta_p}(u'_\ell(\omega))$;
Using Equation (43) compute $\mathrm{ord}_p(u'_\ell(\omega))$.
```

We use the notation $\widetilde{O}$ to ignore polylogarithmic factors in $p$ and $M$.

**Proposition 11.** *Algorithm 2 computes the $p$-unit $u'_\ell(\omega)$ to $M$ digits of $p$-adic precision with time complexity*

$$\widetilde{O}(p^2M^3).$$

*Proof.* The bottleneck of the computation is the evaluation of $I'_{\omega_\gamma,\ell,i}$, for each $(\gamma,i) \in \Gamma_0(\ell) \times [1,\ell-1]$ such that $a_{\gamma,i} \neq 0$. Using Equation (40), we must compute the first

$$M' = M + \lfloor \log(M) \rfloor$$

coefficients of

$$h^{(\ell)}_{\omega_\gamma}\left(t,\ell,i,(\frac{a}{p},\frac{b}{p})\right) \in \mathbb{Q}(\omega)[[t]],$$

for all $a,b \in [0,p-1]^2 \setminus \{(0,0)\}$. This can be done in time

$$O(p^2M'^2\log(M')^2) = \widetilde{O}(p^2M^2),$$

using FFT, as described in Subsection 2.3. The computation of Equation (40) is then carried out in time

$$O(p^2M'^2M\log p) = \widetilde{O}(p^2M^3).$$

Finally, the number of nonzero coefficients $a_{\gamma,i}$ is $O(\log(c))$, which is independent of both $M$ and $p$. □

### 3.2. **Previous Computations of Darmon–Dasgupta and Gross–Stark Units.**

By using the description of $u_\ell(\omega)$ as a $p$-adic integral given in Proposition 8, Dasgupta gave in 2007 an algorithm to compute $u_\ell(\omega)$ in the case $\ell = 4$. Using the results of [Das07], Fleischer and Liu [FL21] wrote SageMath code to compute the units for a given prime $\ell$.

The original calculations of Dasgupta were done up to $O(p^{50})$ $p$-adic precision for $\ell = 4, p = 3, 5, 7, 11$ and $D \leq 500$ while those of Fleischer and Liu were done for $\ell, p \leq 19$, $D \leq 10000$ and up to $O(p^{100})$. Using a similar strategy, Chapdelaine [Cha09] computes $p$-units that live in ray class fields of real quadratic fields with an accuracy of 200 $p$-adic digits for primes $p \leq 17$.

We now describe the strategy given in [Das07] to highlight the main differences with ours. The first step of the strategy is independent of $\omega$. For a given prime $p$ and $\ell \neq p$, one computes specific moments of the measures $\mu_\ell^{DD}\{\infty \to \frac{d}{c}\}$.

In [Das07], Algorithm 4.2 describes how this reduces to the computation of the integrals

$$\text{(i)} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \log_p(Y)\, d\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(X,Y) \quad \text{(ii)} \int_{i+p\mathbb{Z}_p} (t-i)^n\, d\overline{\mu_\ell^{DD}}\left\{\infty \to \frac{d}{c}\right\}(t)$$

$$\text{(iii)} \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} \log_p(X)\, d\mu_\ell^{DD}\left\{\infty \to \frac{d}{c}\right\}(X,Y) \quad \text{(iv)} \int_{\mathbf{P}^1(\mathbb{Q}_p)-\mathbb{Z}_p} t^{-n}\, d\overline{\mu_\ell^{DD}}\left\{\infty \to \frac{d}{c}\right\}(t),$$

for $0 \leq i \leq p-1$ and $0 \leq n \leq M-1$. Here $\overline{\mu_\ell^{DD}}$ denotes a push forward of $\mu_\ell^{DD}$ to $\mathbf{P}^1(\mathbb{Q}_p)$.

We explain in the proof of Proposition 12 how integrals (i) and (iii) can be computed by approximating $\log_p$ with a polynomial. Using the same decomposition of modular symbols as in Equation (45), they also reduce the computation to the case $c = \ell$ (see [Das07] Algorithm 4.1).

They store the values of these integrals. For $p = 13$, $\ell = 5$, and 100-adic precision, the size of the file is 19 MB. The size of the file grows in $\widetilde{O}(pM^2)$.

The final step is to recover the integral defining $\log_p(u_\ell(\omega))$ using integrals (i)–(iv). Since most of the computation time is spent in the first step, which can be very costly (see Section 4), this strategy is efficient when computing for fixed primes $\ell, p \leq 19$ while varying the discriminant $D$.

By contrast, we leverage the generating series approach to compute Darmon–Dasgupta units for much larger primes $p$ (for example $p = 577$, see Section 4) by performing computations one discriminant at a time.

We conclude this description of Dasgupta's algorithm by analyzing its complexity. Comparing with Proposition 11, we observe that our method improves the complexity by a factor of $pM$.

**Proposition 12.** *The algorithm* 4.2 *in* [Das07] *computes integrals* (i)-(iv) *with $M$ digits of $p$-adic precision with a time complexity of*

$$\widetilde{O}(p^3 M^4).$$

*Proof.* We describe the time complexity of the integral (iii), which contributes the dominant complexity among the four. The others are handled in a very similar way.

The main ingredient is the computation of the Dedekind sums

$$D_{s,t}(a/c) := \frac{c^{s-1}}{st} \sum_{h=1}^{c} b_s^\sharp(h/c) b_t^\sharp(ha/c), \tag{48}$$

where $a$ and $c$ are relatively prime integers, and $b_k^\sharp$ denotes the periodic Bernoulli function

$$b_k^\sharp(x) := \begin{cases} 0 & \text{if } k = 1 \text{ and } x \in \mathbb{Z}, \\ k!\, b_k(\{x\}) & \text{otherwise.} \end{cases} \tag{49}$$

The Bernoulli polynomials are viewed with $p$-adic coefficients of precision $M$. Since $b_k^\sharp$ has degree $k$ and coefficients of size $O(M \log p)$, evaluating $b_k^\sharp(x)$ for $x \in \mathbb{Q}_p$ is done in time

$$O(kM \log p).$$

By approximating $\log_p$ with a polynomial of degree $p(M + \lfloor \log(M) \rfloor)$, integral (iii) boils down to computing the moments

$$I_k(a/c) = \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} X^k \, d\mu_\ell^{DD}\{\infty \to \frac{a}{c}\} \tag{50}$$

for $k = 0, \dots, p(M + \lfloor \log(M) \rfloor)$. Using [Das07] Equations (41), $I_k$ is given explicitly by

$$\begin{aligned} I_k(a/c) :=& -12 \sum_{i=0}^{k} \binom{k}{i} \left(\frac{a}{c}\right)^{k-i} (-1)^i \times \\ & \sum_{d \mid \ell} \frac{n_d}{d^i} \left( p^{k-i} D_{k-i+1,i+1}\left(\frac{pa}{c/d}\right) - p^k D_{k-i+1,i+1}\left(\frac{a}{c/d}\right)\right), \end{aligned} \tag{51}$$

where $n_1 = -\ell$ and $n_\ell = 1$. This computation of $I_k$ has complexity

$$O(k^2 M \log p),$$

for a total complexity of

$$O\big((p(M + \lfloor \log(M) \rfloor))^3 M \log p\big) = \widetilde{O}(p^3 M^4).$$

Similarly, integral (i) is computed in $\widetilde{O}(p^2 M^3)$, while integrals (ii) and (iv) contribute $\widetilde{O}(pM^4)$.

□

A way to compute logarithms of Gross–Stark units when they lie in $\mathbb{Q}_p$ rather than $\mathbb{Q}_{p^2}$ is through the computation of values of $p$-adic $L$-functions attached to real quadratic fields. Computations of this type are carried out in [Sla07], [TY13], [Rob15], and [LV22]. Note that, in these constructions of $p$-units, no $\ell$-smoothing is involved.

For their strategy, the authors of [LV22] rely on an algorithm computing the space of overconvergent modular forms. By the main theorem of [DPV21], they construct a $p$-adic modular form whose constant coefficient given in terms of a special value of a $p$-adic $L$-function. Knowing its other coefficients, they write this modular form in the basis given by their algorithm and are able to deduce the value of the constant term.

Using the refinement of [DPV21] given in [DPV24], Damm-Johnsen [DJ24] used a similar idea, instead of the derivative of the $L$-function, they are able to compute directly the $p$-adic logarithm of the unit from which they are able to recover it completely. The computations of Damm-Johnsen are done for primes less than 20 and discriminants up to 10000. An example for $p = 41$ is also given. The code is available there [DJ23]. The precisions used for the calculations can go to $M = 100$.

## 4. NUMERICAL EXAMPLES

The computations of this section were done on the Team Ouragan (INRIA, IMJ–PRG) servers with 128GB of RAM and 32 Intel i9-13900 processors. The code was run using SageMath version 9.5 [The22].

The general strategy to compute $P_{u'_\ell(\omega)}$ is as follows. For each class in the narrow class group of $F$, we find a quadratic form $a_ix^2 + b_ixy + c_iy^2$ with $\ell | a_i$. We pick $\omega_i = \dfrac{-b_i + \sqrt{D}}{2a_i}$, the minimal polynomial over $F$ is then given by

$$P_{u_\ell(\omega)} = P_{u'_\ell(\omega)} = \prod_i (x - u_\ell(\omega_i))).$$

We now give two examples of discriminants for which we compute $u'_\ell(\omega)$ for multiples primes $p$. More tables are available in the author's Github repository [CN26].

**Example 13.** Let $D = 24$ and $\ell = 5$, we take

$$\omega = \frac{\sqrt{D}}{10} + \frac{4}{5}, \quad \gamma_\omega = \begin{pmatrix} 13 & -4 \\ 10 & -3 \end{pmatrix}.$$

In particular, we have $(c, d) = (10, -3)$. As the narrow class group of $F = \mathbb{Q}(\sqrt{24})$ is of order 2, we know that the minimal polynomial $P_{u'_\ell(\omega)}$ of $u'_\ell(\omega)$ with coefficients in $F$ is of degree 2. In particular,

$$P_{u_\ell(\omega)} = (x - u_\ell(\omega))(x - u'_\ell(\omega)) = (x - u_\ell(\omega))(x - u_\ell(\omega)^{-1}).$$

We use SageMath to recognize the coefficient of $x$ of $P_{u_\ell(\omega)}$. The resulting polynomials, along with computation timings, are listed in Tables (1)-(4).

| $p$ | $P_{u_\ell(\omega)}$ | Computation time | Computation time of [FL21] |
|---|---|---|---|
| 7 | $x^2 - \frac{11}{7}x + 1$ | 55s | 1h 2m 25s |
| 11 | $x^2 + (-\frac{3}{11}\sqrt{D} - \frac{7}{11})x + 1$ | 2m 30s | 2h 36m 42s |
| 13 | $x^2 - \frac{1}{13}x + 1$ | 3m 37s | 3h 43m 50s |
| 17 | $x^2 + (-\frac{6}{17}\sqrt{D} + \frac{1}{17})x + 1$ | 6m 39s | 7h 43m 38s |
| 31 | $x^2 + \frac{13}{31}x + 1$ | 25m 15s | x |
| 37 | $x^2 + \frac{26}{37}x + 1$ | 38m 12s | x |
| 41 | $x^2 + (\frac{12}{41}\sqrt{D} + \frac{23}{41})x + 1$ | 45m 9s | x |

TABLE 1. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 100$.

For $p = 7, 11, 13, 17$ we ran the code of [FL21] and gave the running time in the fourth column. By doing computation one discriminant at a time, we are able to compute significantly faster than [FL21] for a fixed discriminant, which allows us to compute for larger primes $p$.

As the prime $p$ grows, we reduce the precision $M$ that we need to recognize the coefficients as elements of $\mathbb{Q}(\omega)$.

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 59 | $x^2 + \frac{82}{59}x + 1$ | 15m 26s |
| 61 | $x^2 - \frac{47}{61}x + 1$ | 14m 1s |
| 79 | $x^2 + \frac{142}{79}x + 1$ | 24m 8s |
| 83 | $x^2 + (\frac{9}{83}\sqrt{D} - \frac{79}{83})x + 1$ | 26m 49s |
| 89 | $x^2 + (-\frac{18}{89}\sqrt{D} + \frac{73}{89})x + 1$ | 31m 17s |
| 103 | $x^2 - \frac{194}{103}x + 1$ | 42m 8s |
| 107 | $x^2 + (-\frac{21}{107}\sqrt{D} - \frac{89}{107})x + 1$ | 45m 37s |
| 109 | $x^2 - \frac{143}{109}x + 1$ | 47m 55s |

TABLE 2. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 50$.

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 521 | $x^2 + (\frac{48}{521}\sqrt{D} - \frac{503}{521})x + 1$ | 37m 5s |
| 541 | $x^2 - \frac{793}{541}x + 1$ | 39m 49s |
| 563 | $x^2 + \frac{226}{563}x + 1$ | 46m 30s |
| 569 | $x^2 + (-\frac{168}{569}\sqrt{D} + \frac{313}{569})x + 1$ | 47m 43s |
| 587 | $x^2 + (\frac{51}{587}\sqrt{D} + \frac{569}{587})x + 1$ | 50m 40s |

TABLE 3. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 10$.

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 7 | $x^2 - \frac{11}{7}x + 1$ | 16h 40m 15s |
| 11 | $x^2 + (-\frac{3}{11}\sqrt{D} - \frac{7}{11})x + 1$ | 47h 59m 45s |

TABLE 4. Minimal polynomial for $D = 24$, $\ell = 5$, $M = 1000$.

We see in Table 3 that we are able to compute the values for large $p$. Our algorithm also allows us to work with very high $p$-adic precision when $p$ is small. In Table 4, we give examples with 1000 $p$-adic digit accuracy, a comparable precision to similar computations done in the archimedean setting, see for example [BCG23].

The polynomials in these tables correspond to the same extension $H$ of $F = \mathbb{Q}(\sqrt{24})$. Here

$$H = F(j) = \mathbb{Q}(\sqrt{24}, j) = \mathbb{Q}(\sqrt{24} + j), \tag{52}$$

where $j = e^{\frac{2i\pi}{3}}$. The field $H$ is, as expected, the narrow class field of $F$.

**Example 14.** Let $D = 156$ and $\ell = 5$, take

$$\omega_1 = \frac{\sqrt{D}}{20} + \frac{3}{10}, \ \gamma_{\omega_1} = \begin{pmatrix} 37 & 12 \\ 40 & 13 \end{pmatrix}, \ \omega_2 = \frac{\sqrt{D}}{10} - \frac{2}{5}, \ \gamma_{\omega_2} = \begin{pmatrix} 17 & 28 \\ 20 & 33 \end{pmatrix}. \tag{53}$$

This time we have $(c_1, d_1) = (40, 13)$ and $(c_2, d_2) = (20, 33)$. The narrow class group of $F = \mathbb{Q}(\sqrt{156})$ is of order 4, $u'_\ell(\omega_1)$ and $u'_\ell(\omega_2)$ are conjugate, their minimal polynomial over $F$ is given by

$$P_{u_\ell(\omega)} = (x - u'_\ell(\omega_1))(x - u'_\ell(\omega_1)^{-1})(x - u'_\ell(\omega_2))(x - u'_\ell(\omega_2)^{-1}) \in F[x].$$

The polynomial $P_{u_\ell(\omega)}$ is palindromic, we write only the first coefficients. All the polynomials in these tables generate the same abelian extension $H$ of $F$. The field $H$ is, as expected, the narrow class field of $F$. It is generated over $\mathbb{Q}$ by the polynomial

$$\begin{aligned} P(x) =& 214358881 \times x^8 - 904462416 \times x^7 + 2001359602 \times x^6 \\ & - 3077062560 \times x^5 + 3538768611 \times x^4 - 3077062560 \times x^3 \\ & + 2001359602 \times x^2 - 904462416 \times x + 214358881. \end{aligned} \tag{54}$$

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 11 | $x^4+(-\frac{24}{11^3}\sqrt{D}-\frac{2808}{11^3})x^3$ $+(\frac{6}{11^2}\sqrt{D}+\frac{36137}{11^4})x^2+\dots$ | 8m 56s |
| 17 | $x^4+(-\frac{644}{17^3}\sqrt{D}+\frac{2898}{17^3})x^3$ $+(-\frac{36}{17^2}\sqrt{D}+\frac{140387}{17^4})x^2+\dots$ | 21m 28s |

Table 5. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 100$.

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 73 | $x^4-\frac{66332}{73^3}x^3-\frac{43440954}{73^4}x^2+..$ | 1h 29m 37s |
| 79 | $x^4+\frac{1417444}{79^3}x^3+\frac{147583686}{79^4}x^2+..$ | 1h 44m 32s |
| 83 | $x^4+(\frac{3128}{83^3}\sqrt{D}+\frac{150696}{83^3})x^3$ $+(\frac{714}{83^2}\sqrt{D}+\frac{32851337}{83^4})x^2+..$ | 1h 56m 8s |
| 97 | $x^4+\frac{324142}{97^3}x^3+\frac{95350611}{97^4}x^2+..$ | 2h 38m 28s |
| 101 | $x^4+(-\frac{263228}{101^3}\sqrt{D}+\frac{56406}{101^3})x^3$ $+(-\frac{84}{101^2}\sqrt{D}+\frac{457391987}{101^4})x^2+..$ | 2h 53m 57s |
| 103 | $x^4-\frac{3631292}{103^3}x^3+\frac{524542086}{103^4}x^2+..$ | 2h 59m 50s |
| 109 | $x^4+\frac{1338766}{109^3}x^3+\frac{48285291}{109^4}x^2+..$ | 3h 25m 19s |

Table 6. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 50$.

| $p$ | $P_{u_\ell(\omega)}$ | Computation time |
|---|---|---|
| 521 | $x^4 + \frac{59731644}{521^3}x^3 - \frac{75691649914}{521^4}x^2 + ..$ | 2h 53m 37s |
| 523 | $x^4 + \frac{43862518}{523^3}x^3 - \frac{80826503229}{523^4}x^2 + ..$ | 2h 40m 43s |
| 541 | $x^4 + \frac{95348734}{541^3}x^3 + \frac{27184995291}{541^4}x^2 + ..$ | 2h 49m 45s |
| 547 | $x^4 + \frac{254172908}{547^3}x^3 + \frac{149879597286}{547^4}x^2 + ..$ | 2h 55m 42s |
| 569 | $x^4 + (\frac{32989928}{569^3}\sqrt{D} + \frac{66413934}{569^3})x^3$<br>$+(\frac{23256}{569^2}\sqrt{D} + \frac{278087052707}{569^4})x^2 + ..$ | 3h 8m 9s |
| 571 | $x^4 - \frac{593232794}{571^3}x^3 + \frac{467704609011}{571^4}x^2 + ..$ | 3h 10m 6s |
| 577 | $x^4 + \frac{46621582}{577^3}x^3 + \frac{211173573171}{577^4}x^2 + ..$ | 3h 15m 21s |

Table 7. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 10$.

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